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Strong Summability in Fréchet Spaces with Applications to Fourier Series

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This paper examines strong Cesàro summability and strong Cesàro sectional boundedness of order $1 \leqslant r < \infty$ in Banach and Fréchet spaces E. The major result shows these topological properties of E to be equivalent to multiplier properties of the form $E = (dv_r \cap c_0) \cdot E$ and $E = dv_r \cdot E$, where dv_r is the space of sequences of dyadic variation of order r defined in this paper. These multiplier results show that several classical spaces of Fourier series have these properties. This introduces a new form of convergence in norm for Fourier series. The space $L^1_{2\pi}$, for example, has strong Cesàro summability of all orders $1 \leqslant r < \infty$. Fejér's Theorem states that for all $f \in L^1_{2\pi}$, $(1/(n+1))\|\sum_{k=0}^n s^k f - f\|_{L^1} = o(1)$, $(n \to \infty)$, where $s^k f$ is the kth partial sum of the Fourier series of f; since the dual of $L^1_{2\pi}$ is $L^\infty_{2\pi}$, this is equivalent to $\sup_{\|g\|_{L^\infty} \leqslant 1} (1/(n+1))|\sum_{k=0}^n \int_0^{2\pi} g \cdot (s^k f - f)| = o(1)$, $(n \to \infty)$. As a consequence of strong Cesàro summability, the absolute value can be taken inside the summation and raised to any power $1 \leqslant r < \infty$. Namely, for all $f \in L^1_{2\pi}$,

$$\sup_{\|g\|_{L^\infty} \leqslant 1} \frac{1}{n+1} \sum_{k=0}^{n} \left| \int_0^{2\pi} g \cdot (s^k f - f) \right|^r = o(1) \qquad (n \to \infty).$$

The supremum, however, cannot be taken inside the summation.

1. INTRODUCTION

A Fréchet space is a complete metrizable locally convex space; for example, every Banach space is a Fréchet space. Consider a Fréchet space E with a total biorthogonal sequence $\{e^k, f_j\}$ [1]. That is,

$$e^k \in E \qquad \text{for all} \quad k; \tag{1.A}$$

$$f_j \in E' \qquad \text{(the space of continuous linear functionals) for all } j; \tag{1.B}$$

$$f_j(e^k) = \delta_{jk} \qquad \text{(Kronecker } \delta\text{) for all } k \text{ and } j; \tag{1.C}$$

$$f_j(x) = 0 \qquad \text{for all } j \text{ implies } x = 0. \tag{1.D}$$

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Generally we assume that the indices k and j range over the nonnegative integers but when discussing Fourier series of 2π-periodic functions the indices will range over all integers. Each x in E can be identified with the sequence $\hat{x}=(x_0, x_1, x_2, ...)$ where $x_j=f_j(x)$. Let $\hat{E}=\{\hat{x} \mid x \in E\}$. E and $\hat{E}$ are isometric and isomorphic if, for each defining seminorm p_E on E, we define $p_{\hat{E}}(\hat{x})=p_E(x)$. If E is a Banach space we define $\|\hat{x}\|_{\hat{E}}=\|x\|_E$. By conditions (1.B) and (1.C), $\hat{E}$ has continuous coordinate functionals. Such a Fréchet (respectively Banach) sequence space is called an FK-space (respectively, BK-space). By condition (1.A), $\hat{E}$ contains the space of finite sequences

$$\phi := \{x=(x_k) \mid x_k=0 \text{ except for finitely many } k\}.$$

For simplicity, most theorems in this paper will be stated for FK-spaces (that is, $E=\hat{E}$ where, for each k, e^k is the sequence with 1 in the kth position and 0 elsewhere); however, when considering function spaces it will often be more convenient to work directly on E instead of the corresponding FK-space $\hat{E}$.

An element x in E has the property of sectional convergence (denoted AK) in E if the sections $s^n x := x_0 e^0 + x_1 e^1 + \cdots + x_n e^n$ converge to x (as $n \to \infty$) with respect to the topology of E. In case the biorthogonal sequence ranges over all integers, we define $s^n x := \sum_{|k| \leqslant n} x_k e^k$ for $n=0, 1, 2, ...$. More generally, an element x, not necessarily in E, has the property of sectional boundedness (denoted AB) in E if the sections $s^n x$ are bounded in E. Similarly an element x in E has the property of Cesàro sectional convergence (denoted σK) in E if the Cesàro sections $\sigma^n x := (s^0 x + \cdots + s^n x)/(n+1)$ converge to x (as $n \to \infty$), with respect to the topology of E. This is equivalent to

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} (s^k x - x) = 0.$$

An element x, not necessarily in E, has the property of Cesàro sectional boundedness (denoted σB) in E if $\sup_n p(\sigma^n x) < \infty$ for all continuous seminorms p on E.

Let $1 \leqslant r < \infty$. Section 2 contains basic definitions and introduces the properties of strong Cesàro summability of order r (denoted $[\sigma K]_r$) and strong Cesàro boundedness of order r (denoted $[\sigma B]_r$) in Fréchet spaces. These properties are stronger than σK and σB, respectively, but are weaker than AK and AB, respectively. Section 3 contains general results on strong Cesàro summability and strong Cesàro boundedness in Fréchet spaces. In Section 4 specific spaces are considered; namely the convergence fields H_r and boundedness domains B_r of the strong Cesàro summability methods, and their spaces of convergence factors dv_r and $dv_r \cap c_0$. In Section 5 we show the equivalence of the properties $[\sigma B]_r$ and $[\sigma K]_r$ to multiplier properties with respect to the spaces dv_r and $dv_r \cap c_0$. In particular, a Fréchet space E containing ϕ has the property $[\sigma B]_r$ if and only if $E = dv_r \cdot E$, and it has the property $[\sigma K]_r$ if and only if $E = (dv_r \cap c_0) \cdot E$. In Section 6 we consider function spaces and show how these multiplier results can be used to obtain a new form of convergence for Fourier series. For example, we show that the spaces $L^p_{2\pi}$ $(1 \leqslant p < \infty)$ and $C_{2\pi}$ (2π-periodic continuous functions) have the property $[\sigma K]_r$ for all r and the spaces $L^\infty_{2\pi}$ and $M_{2\pi}$ (2π-periodic Radon measures) have the property $[\sigma B]_r$ for all r. Fejér's theorem for $L^1_{2\pi}$ is equivalent to the property σK but, for all r, the property $[\sigma K]_r$ is stronger.

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2. Definitions

Let E be an FK-space containing ϕ. By the Hahn–Banach theorem each continuous seminorm p can be expressed in the form

$$p(x) = \sup_{f \in A_p} |f(x)| \tag{2.A}$$

for some subset A_p of E'. Thus an element x of E has the property σK if

$$\lim_{n \to \infty} \sup_{f \in A_p} \left| f\left(\frac{1}{n+1} \sum_{k=0}^{n} (s^k x - x) \right) \right| = 0$$

for every continuous seminorm p. Let $1 \leqslant r < \infty$. We define the property of strong Cesàro summability of order r (denoted $[\sigma K]_r$) for $x \in E$ by

$$\lim_{n \to \infty} \sup_{f \in A_p} \frac{1}{n+1} \sum_{k=0}^{n} |f(s^k x - x)|^r = 0 \tag{2.B}$$

for every continuous seminorm p. Similarly a formal expansion $x = \sum_{k=0}^\infty x_k e^k$ with $\hat{x} = (x_0, x_1, ...)$, but $\hat{x}$ not necessarily in $\hat{E}$, has strong Cesàro boundedness of order r (denoted $[\sigma B]_r$) in E if

$$p_r(x) := \sup_n \sup_{f \in A_p} \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |f(s^k x)|^r \right\}^{1/r} < \infty \tag{2.C}$$

for each continuous seminorm p. If E is a Banach space, we write $\|x\|_r$ instead of $p_r(x)$.

A continuous seminorm p on E does not uniquely determine the set A_p. For now, we will assume that

$$A_p = \{f \in E' \mid |f| \leqslant p\}; \tag{2.D}$$

however, a proper subset of (2.D) is often more natural. For example, if $E = C_{2\pi}$ with norm $\|g\|^{\infty} = \sup_y |g(y)|$, it is natural to choose $A_{\|\cdot\|^\infty}$ to be the set of f_y defined by $f_y(g) = g(y)$, $0 \leqslant y \leqslant 2\pi$. Theorems (3.1) and (3.3) (or more explicitly, (5.1)) show that the choice of A_p does not affect the definitions of $[\sigma K]_r$ and $[\sigma B]_r$. We write

$$E_{[\sigma K]_r} := \{x \in E \mid x \text{ has the property } [\sigma K]_r \text{ in } E\},$$

$$E_{[\sigma B]_r} := \left\{x = \sum_{k=0}^{\infty} x_k e^k \,\middle|\, x \text{ has the property } [\sigma B]_r \text{ in } E\right\},$$

$$E_{AD} := \{x \in E \mid x \text{ is in the closure of } \phi \text{ in } E\},$$

and similarly for E_{AK}, E_{AB}, $E_{\sigma K}$, and $E_{\sigma B}$. Note that the spaces $E_{[\sigma B]_r}$, $E_{\sigma B}$, and E_{AB} need not be subsets of E.

We say that E has the property $[\sigma K]_r$ if $E = E_{[\sigma K]_r}$; E has the property $[\sigma B]_r$ if $E \subset E_{[\sigma B]_r}$; E has the property AD if $E = E_{AD}$; etc.

Hölder's inequality applied to (2.C) shows that $p_r(x) \leqslant p_s(x)$ for $1 \leqslant r \leqslant s < \infty$. Similarly it can be shown that $[\sigma K]_s$ implies $[\sigma K]_r$ and $[\sigma B]_s$ implies $[\sigma B]_r$, if $1 \leqslant r \leqslant s < \infty$. It is clear from the definitions that $[\sigma K]_r$ implies both σK and $[\sigma B]_r$. By an argument similar to that showing that ordinary summability implies Cesàro summability, it can be shown that, for all r, the property AK in E implies

$$\lim_{n \to \infty} \frac{1}{n+1} \sum_{k=0}^{n} p(s^k x - x)^r = 0 \quad \text{for all continuous seminorms } p. \tag{2.E}$$

This in turn implies $[\sigma K]_r$. Similarly the property AB implies

$$\sup_n \left\{\frac{1}{n+1} \sum_{k=0}^{n} p(s^k x)^r\right\}^{1/r} < \infty \quad \text{for all continuous seminorms } p \tag{2.F}$$

which implies $[\sigma B]_r$. The converse implications are not true as shown by examples in (4.4) and (6.3).

It will be shown in Theorem 3.1 that an FK-space E has the property $[\sigma K]_r$ if and only if it has the properties AD and $[\sigma B]_r$.

The convergence field for strong Cesàro summability of order r is denoted by

$$H_r := \left\{ x = (x_k) \mid \exists s \in \mathbb{C} \ni \lim_n \frac{1}{n+1} \sum_{k=0}^{n} |X_k - s|^r = 0 \right\},$$

where $X_k = x_0 + \cdots + x_k$ (or $X_k = \sum_{|j| \leqslant k} x_j$ if the index k ranges over all integers) and the boundedness domain for strong Cesàro summability of order r is

$$B_r := \left\{ x = (x_k) \mid \|x\|_{B_r} := \sup_n \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |X_k|^r \right\}^{1/r} < \infty \right\}.$$

Below we list some useful BK-spaces and their norms. We use the notation $\Delta y_k = y_k - y_{k+1}$, $\Delta^2 y_k = \Delta y_k - \Delta y_{k+1}$, $\max_{2^n} = \max_{2^n \leqslant k < 2^{n+1}}$, and $\sum_{2^n} = \sum_{k=2^n}^{2^{n+1}-1}$. If the index k ranges over the integers we assume that $y_k = y_{-k}$ for the spaces bv, dv_r, and q listed below. The *space of bounded sequences* is

$$l^\infty = \{ x = (x_k) \mid \|x\|_\infty := \sup |x_k| < \infty \}$$

and the *space of null sequences*

$$c_0 = \{ x = (x_k) \mid \lim x_k = 0 \}$$

is a closed subspace of l^∞. The *space of sequences of bounded variation* is

$$bv = \left\{ y = (y_k) \mid \|y\|_{bv} := \sum_{k=0}^{\infty} |\Delta y_k| + \sup |y_k| < \infty \right\}.$$

The *spaces of sequences of dyadic variation* are defined as

$$dv = dv_1 = \left\{ y = (y_k) \mid \|y\|_{dv} := \sum_{n=0}^{\infty} 2^n \max_{2^n} |\Delta y_{k-1}| + \sup |y_k| < \infty \right\},$$

$$dv_r = \left\{ y = (y_k) \mid \|y\|_{dv_r} := \sum_{n=0}^{\infty} 2^{n/r} \left(\sum_{2^n} |\Delta y_{k-1}|^s\right)^{1/s} + \sup |y_k| < \infty \right\}$$

for $1 < r < \infty$ and $1/r + 1/s = 1$. It is natural to define $dv_\infty = bv$. The *space of bounded quasiconvex sequences* is

$$q = \left\{ y = (y_k) \mid \|y\|_q := \sum_{k=0}^{\infty} (k+1)\, |\Delta^2 y_k| + \sup |y_k| < \infty \right\}.$$

We have $q \subset dv \subset dv_r \subset dv_s \subset bv$ for $1 \leqslant r \leqslant s \leqslant \infty$ [6, 9].

3. General Results

Zeller has shown that an FK-space E containing ϕ has the property AK if and only if it has the properties AB and AD. A corresponding theorem for σK was proved in [3]. Using a similar $\varepsilon/3$ argument we obtain it for the property $[\sigma K]_r$:

(3.1) Theorem. *Let $1 \leqslant r < \infty$. An FK-space E containing ϕ has the property $[\sigma K]_r$ if and only if it has the properties $[\sigma B]_r$ and AD.*

Proof. Let $x \in E$ have the property $[\sigma K]_r$ and let p be a continuous seminorm on E. The property $[\sigma B]_r$ follows immediately. For each n, $\sigma^n x \in \phi$ and $p(\sigma^n x - x) = \sup_{f \in A_p} |f(1/(n+1)) \sum_{k=0}^{n} (s^k x - x))| \leqslant \sup_{f \in A_p} (1/(n+1)) \sum_{k=0}^{n} |f(s^k x - x)| \to 0$ as $n \to \infty$. Thus x has the property AD. Conversely, suppose E has the properties AD and $[\sigma B]_r$. Since for all $x \in E$ we have $p_r(x) = \sup_n \sup_{f \in A_p} \{(1/(n+1)) \sum_{k=0}^{n} |f(s^k x)|^r\}^{1/r} < \infty$, the seminorm p_r is lower semicontinuous and hence continuous on E (Theorem 7.1.1 of [7]). Let $x \in E$ and $\varepsilon > 0$. Since E has the property AD we can find $y \in \phi$ such that $p(x-y) < \varepsilon/3$ and $p_r(x-y) < \varepsilon/3$. We can choose n sufficiently large so that $\{(1/(n+1)) \sum_{k=0}^{n} p(s^k y - y)^r\}^{1/r} < \varepsilon/3$. Let $|f| \leqslant p$. Then by Minkowski's inequality

$$\begin{aligned}
&\left\{ \frac{1}{n+1} \sum_{k=0}^{n} |f(s^k x - x)|^r \right\}^{1/r} \\
&\quad \leqslant \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |f(s^k x - s^k y)|^r \right\}^{1/r} + \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |f(s^k y - y)|^r \right\}^{1/r} \\
&\qquad + \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |f(y - x)|^r \right\}^{1/r} \\
&\quad < p_r(x-y) + \left\{ \frac{1}{n+1} \sum_{k=0}^{n} p(s^k y - y)^r \right\}^{1/r} + p(y-x) < \varepsilon.
\end{aligned}$$

Taking the supremum over all $|f| \leqslant p$, we see that x has the property $[\sigma K]_r$. ▮

(3.2) COROLLARY. *Let* $1 \leqslant r < \infty$. *If an FK-space containing* ϕ *has the property* $[\sigma B]_r$, *then* $E_{[\sigma K]_r} = E_{AD}$.

Proof. E_{AD}, being a closed subspace of E, is an FK-space with the properties AD and $[\sigma B]_r$. By (3.1) we have $E_{AD} \subset E_{[\sigma K]_r}$. The inclusion $E_{[\sigma K]_r} \subset E_{AD}$ is shown in the proof of (3.1). ▮

For FK-spaces E and F, define

$$E^{\varphi} = \{ y = (y_k) \mid \text{for some } f \in E',\ y_k = f(e^k) \}$$

and

$$E^F = \{ y = (y_k) \mid x \cdot y = (x_k y_k) \in F\ \forall x \in E \}.$$

If E is a BK-space, then E^{φ} can be identified with the dual space of E_{AD} [5] and is thus a BK-space under the dual norm of E_{AD}. However, for an FK-space E, the space E^{φ} need not be an FK-space (the space $E = \omega$ of all complex valued sequences is such an example). The proof of the following theorem is consequently more difficult than in the BK-space case.

(3.3) THEOREM. *Let* $1 \leqslant r < \infty$ *and let* E *be an FK-space containing* ϕ. *A sequence* x *has the property* $[\sigma B]_r$ *with respect to* E *if and only if*

$$s_f(x) := \sup_n \frac{1}{n+1} \sum_{k=0}^{n} |f(s^k x)|^r < \infty$$

for every $f \in E'$.

Proof. The proof of ($\Rightarrow$) is obvious. For the converse, we use a uniform boundedness argument. Suppose $s_f(x) < \infty$ for all $f \in E'$. Let p be a continuous seminorm on E and let E_p be the space E under the locally convex topology defined by the single seminorm p. Then E'_p is a Banach space with unit ball $U = \{ f \in E'_p \mid |f| \leqslant p \}$. (Actually E^{φ}_p is a BK-space with $E^{\varphi}_p \subset E^{\varphi}$). Let $B = \{ f \in E'_p \mid s_f(x) \leqslant 1 \}$. B is clearly absolutely convex and closed. B is radial (absorbing) since $s_f(x) < \infty$ for all $f \in E'_p$. Thus B is a barrel in E'_p. Since E'_p is barreled, B is a neighborhood of 0. Thus there exists an $N > 0$ such that $(1/N)\, U \subset B$. Then $\sup_{|f| \leqslant p} s_f(x) \leqslant N$. ▮

(3.4) COROLLARY. *Let* $1 \leqslant r < \infty$ *and let* E *be an FK-space containing* ϕ. *Then*

$$E_{[\sigma B]_r} = (E^{\varphi})^{B_r}.$$

(3.5) COROLLARY. *Let $1 \leqslant r < \infty$ and let E be an FK-space containing ϕ. Then*

(a) *E has the property $[\sigma B]_r$ if and only if $E^{\varphi} = E^{B_r}$;*

(b) *E has the property $[\sigma B]_r$ if and only if $E_{[\sigma B]_r} = (E^{B_r})^{B_r}$;*

(c) *if E has the property $[\sigma K]_r$, then $E^{\varphi} = E^{H_r}$.*

Proof. Let $(X \cdot Y)_k = x_0 y_0 + \cdots + x_k y_k$ and define

$$E^{\sigma b} := \left\{ y = (y_k) \;\middle|\; \sup_n \frac{1}{n+1} \left| \sum_{k=0}^{n} (X \cdot Y)_k \right| < \infty \text{ for all } x \in E \right\}.$$

It was shown in [4, Theorem 3] that for any FK-space E containing ϕ we have $E^{\sigma b} \subset E^{\varphi}$. Clearly $E^{B_r} \subset E^{\sigma b} \subset E^{\varphi}$. Thus (a) follows immediately from (3.3). Statement (b) follows from (a), (3.4), and $E \subset (E^{B_r})^{B_r}$. Statement (c) is a consequence of $E^{H_r} \subset E^{B_r} \subset E^{\varphi}$ and the definition of $[\sigma K]_r$. ▮

(3.6) THEOREM. *Let $1 \leqslant r < \infty$. If E is an FK-space containing ϕ generated by a set of seminorms P, then $E_{[\sigma B]_r}$ is an FK-space under the topology generated by the set of seminorms $\{p_r \mid p \in P\}$.*

Proof. It is shown in [3] that $E_{\sigma B}$ is an FK-space under the seminorms $q_p(x) := \sup_n p((1/(n+1)) \sum_{k=0}^{n} s^k x)$. We have $q_p \leqslant p_1$ by the definition of p_1; also $p_1 \leqslant p_r$ by Hölder's inequality. Since $E_{[\sigma B]_r} = \{x \in E_{\sigma B} \mid p_r(x) < \infty \; \forall p \in P\}$ it follows from Garling's Theorem [10, p. 998] that $E_{[\sigma B]_r}$ is an FK-space. ▮

(3.7) *Remark.* We will show in (5.3) that for every FK-space E containing ϕ, the space $E_{[\sigma B]_r}$ always has the property $[\sigma B]_r$. However, an FK-space E (containing ϕ) with the property $[\sigma B]_r$ need not be a closed subspace of $E_{[\sigma B]_r}$. Thus if the topology of E is generated by a set of seminorms P, the topology generated by the set of seminorms $\{p_r \mid p \in P\}$ need not make E an FK-space.

(3.8) *Remark.* If E is an FK-space containing ϕ, it can be shown that the set of all sequences x (not restricted to those belonging to E) which satisfy condition (2.F) forms an FK-space. The proof is similar to that of (3.3). It can even be shown that this FK-space satisfies the condition (2.F).

4. CONVERGENCE FIELDS AND CONVERGENCE FACTORS

In this section we look at the convergence fields H_r and boundedness domains B_r of the strong Cesàro summability methods, and at their spaces of convergence factors dv_r and $dv_r \cap c_0$. These spaces are of interest in themselves. Moreover, the properties $[\sigma B]_r$ and $[\sigma K]_r$ of these spaces are important in the proofs of multiplier results in the next section.

Maddox [13, p. 101] has observed that H_r is a BK-space under the norm

$$\|x\|_{B_r} = \sup_m \left\{ \frac{1}{m+1} \sum_{j=0}^{m} |X_j|^r \right\}^{1/r},$$

where $X_j = x_0 + \cdots + x_j$. Under the same norm B_r is also a BK-space; this can easily be shown using Garling's Theorem [10, p. 998].

(4.1) THEOREM. *Let $1 \leqslant r < \infty$. Consider the BK-spaces H_r and B_r under the norm $\|\cdot\|_{B_r}$.*

(a) *H_r satisfies condition* (2.E);

(b) *B_r satisfies condition* (2.F);

(c) $H_r = (B_r)_{[\sigma K]_r}$;

(d) $(H_r)_{[\sigma B]_r} = (B_r)_{[\sigma B]_r} = (H_r)_{\sigma B} = (B_r)_{\sigma B} = B_r$.

Proof. (a) Let $x \in H_r$ and $\varepsilon > 0$. We show that $(1/(n+1)) \sum_{k=0}^{n} \|s^k x - x\|_{B_r}^r \to 0$. Since $x \in H_r$, there exists a complex number s such that $(1/(n+1)) \sum_{k=0}^{n} |X_k - s|^r \to 0$. By changing the value of x_0 we may assume $s = 0$. Choose N such that, for all $n > N^2$, $\{(1/(n+1)) \sum_{k=0}^{n} |X_k|^r\}^{1/r} < \varepsilon/3$ and $((N+1)/(n+1))^{1/r} \|x\|_{B_r} < \varepsilon/3$. Using Minkowski's inequality we have for $n > N^2$

$$\begin{aligned}
&\left\{ \frac{1}{n+1} \sum_{k=0}^{n} \|s^k x - x\|_{B_r}^r \right\}^{1/r} \\
&= \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \sup_{m>k} \frac{1}{m+1} \sum_{j=k+1}^{m} |X_j - X_k|^r \right\}^{1/r} \\
&\leqslant \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \sup_{m>k} \frac{1}{m+1} \sum_{j=k+1}^{m} |X_j|^r \right\}^{1/r} \\
&\quad + \left\{ \frac{1}{n+1} \sum_{k=0}^{n} \sup_{m>k} \frac{1}{m+1} \sum_{j=k+1}^{m} |X_k|^r \right\}^{1/r} \\
&\leqslant \left\{ \frac{1}{n+1} \sum_{k=0}^{N} \|x\|_{B_r}^r \right\}^{1/r} \\
&\quad + \left\{ \frac{1}{n+1} \sum_{k=N+1}^{n} \sup_{m>k>N} \frac{1}{m+1} \sum_{j=0}^{m} |X_j|^r \right\}^{1/r} \\
&\quad + \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |X_k|^r \right\}^{1/r} \\
&< \left(\frac{N+1}{n+1} \right)^{1/r} \|x\|_{B_r} + \left\{ \frac{1}{n+1} \sum_{k=N+1}^{n} \left(\frac{\varepsilon}{3} \right)^r \right\}^{1/r} + \frac{\varepsilon}{3} < \varepsilon.
\end{aligned}$$

(b) It is readily shown that

$$\frac{1}{n+1}\sum_{k=0}^{n}\|s^k x\|_{B_r}^r \leqslant \frac{1}{n+1}\sum_{k=0}^{n}\sup_{k<m}\left\{\|x\|_{B_r}^r, \frac{k+1}{m+1}\|x\|_{B_r}^r + \left(1-\frac{k+1}{m+1}\right)|X_k|^r\right\} \leqslant 2\|x\|_{B_r}^r.$$

(c) Since H_r is a closed subspace of B_r with the property $[\sigma K]_r$, $H_r = (B_r)_{AD}$. By (3.2) we have $H_r = (B_r)_{[\sigma K]_r}$.

(d) By (b) we have $B_r \subset (B_r)_{[\sigma B]_r}$. Since H_r is a closed subspace of B_r, we have $(H_r)_{[\sigma B]_r} = (B_r)_{[\sigma B]_r} \subset (H_r)_{\sigma B} = (B_r)_{\sigma B}$. Finally we show $(B_r)_{\sigma B} \subset B_r$ by showing $\|x\|_{B_r} \leqslant \sup_m \|\sigma^m x\|_{B_r}$. Let $\varepsilon > 0$. For each n, we can find m such that $|\sum_{j=0}^k x_j| < |\sum_{j=0}^k (1 - j/(m+1))\, x_j| + \varepsilon$ for $k = 0, ..., n$. Then

$$\left\{\frac{1}{n+1}\sum_{k=0}^{n}|X_k|^r\right\}^{1/r} < \left\{\frac{1}{n+1}\sum_{k=0}^{n}\left(\left|\sum_{j=0}^{k}\left(1-\frac{j}{m+1}\right)x_j\right| + \varepsilon\right)^r\right\}^{1/r}$$
$$\leqslant \left\{\frac{1}{n+1}\sum_{k=0}^{n}\left|\sum_{j=0}^{k}\left(1-\frac{j}{m+1}\right)x_j\right|^r\right\}^{1/r} + \varepsilon$$
$$\leqslant \|\sigma^m x\|_{B_r} + \varepsilon \leqslant \sup_m \|\sigma^m x\|_{B_r} + \varepsilon.$$

Thus $\|x\|_{B_r} = \sup_n\{(1/(n+1))\sum_{k=0}^n |X_k|^r\}^{1/r} \leqslant \sup_m \|\sigma^m x\|_{B_r}$. ▮

Let $\sigma s = \{x = (x_k) \mid \lim_n (1/(n+1)) \sum_{k=0}^n X_k$ exists$\}$ be the series-sequence convergence field of the Cesàro method of order 1. Jackson [11] has shown that

$$H_r^{H_r} = H_r^{\sigma s} = dv_r \tag{4.A}$$

as well as some more general multiplier results. Maddox [12] had earlier shown that $H_r^{\varphi} = dv_r$ (see also [2]). The identities (4.A) thus also follow from (4.1) and (3.5).

(4.2) THEOREM. *Let* $1 \leqslant r < \infty$. *Then* $(dv_r)_{[\sigma B]_r} = (dv_r)_{\sigma B} = dv_r$.

Proof. Since H_r is a closed subspace of B_r, we have $H_r^{\varphi} = B_r^{\varphi} = dv_r$. Since H_r has the property σK, we have $H_r^{\varphi} = H_r^{\sigma s}$ [3]. Thus $dv_r^{B_r} = (B_r^{\varphi})^{\varphi} =$

$(B_r)_{[\sigma B]_r} = B_r$ and $dv_r^{\varphi} = (B_r^{\varphi})^{\varphi} = (B_r)_{\sigma B} = B_r$ [5]. Then $dv_r \cdot dv_r^{\varphi} \subset B_r$ (actually $dv_r \cdot dv_r^{\varphi} = B_r$ since $(1, 1, 1, ...) \in dv_r$). Hence $dv_r \subset (dv_r^{\varphi})^{B_r} = (dv_r)_{[\sigma B]_r} \subset (dv_r)_{\sigma B}$. Conversely, $(dv_r)_{\sigma B} = (H_r^{\varphi})_{\sigma B} \subset H_r^{\varphi} = dv_r$ by [4, Proposition 1]. ▌

Using (3.5)(b) we can now add the following to the multiplier identities (4.A):

$$H_r^{B_r} = B_r^{B_r} = dv_r. \tag{4.B}$$

(4.3) THEOREM. *Let* $1 \leqslant r < \infty$. *Then* $(dv_r)_{[\sigma K]_r} = dv_r \cap c_0$.

Proof. By (3.2), $(dv_r)_{[\sigma K]_r} = (dv_r)_{AD}$. Since $dv_r \subset l^{\infty}$, we have $(dv_r)_{AD} \subset l^{\infty}_{AD} = c_0$. Thus $(dv_r)_{[\sigma K]_r} \subset dv_r \cap c_0$. Conversely, let $y \in dv_r \cap c_0$ and $\sigma^n y = (1/(n+1)) \sum_{k=0}^{n} s^k y = \sum_{k=0}^{n} (1 - k/(n+1))\, y_k e^k$. We show $y \in (dv_r)_{AD}$ by showing $\lim_n \|\sigma^{2^n} y - y\|_{dv_r} = 0$. Let $\varepsilon > 0$ and choose M such that $|y_k| < \varepsilon$ whenever $k > 2^M$ and $\sum_{N=M+1}^{\infty} 2^{N/r} (\sum_{2^N} |\Delta y_k|^{r'})^{1/r'} < \varepsilon$. Let $t = 2^n$ and $n > M$. Then $\|\sigma^t y - y\|_{dv_r} = \|\sigma^t y - y\|_{\infty} + \sum_{N=0}^{M-1} 2^{N/r} (\sum_{2^N} |(k/(t+1))\, y_{k-1} - ((k+1)/(t+1)) y_k|^{r'})^{1/r'} + \sum_{N=M}^{n-1} 2^{N/r} (\sum_{2^N} |(k/(t+1))\, y_{k-1} - ((k+1)/(t+1))\, y_k|^{r'})^{1/r'} + \sum_{N=n}^{\infty} 2^{N/r} (\sum_{2^N} |\Delta y_{k-1}|^{r'})^{1/r'} = S_1 + S_2 + S_3 + S_4$. Since $y \in c_0$ we have $S_1 = o(1)$. Clearly $S_2 = o(1)$ and $S_4 < \varepsilon$. Finally $S_3 = \sum_{N=M}^{n-1} 2^{N/r} (\sum_{2^N} |(k/(t+1))\, \Delta y_{k-1} + (1/(t+1))\, y_k|^{r'})^{1/r'} \leqslant \sum_{N=M}^{n-1} 2^{N/r} (\sum_{2^N} |\Delta y_{k-1}|^{r'})^{1/r'} + (\varepsilon/(t+1)) \sum_{N=M}^{n-1} 2^{N/r} \cdot 2^{(N+1)/r'} < \varepsilon + (\varepsilon/(t+1)) \sum_{N=M}^{n-1} 2^{N+1} < \varepsilon + (\varepsilon/(t+1))\, 2^n < 2\varepsilon$. ▌

(4.4) *Remark.* The spaces dv_r do not satisfy (2.F) as can be seen by considering the sequence $(1, 1, 1, ...)$. It can be shown that y in dv_r satisfies (2.F) if and only if

$$\sum_{2^n} |y_{k-1}| = O(2^{n/s}), \qquad \frac{1}{r} + \frac{1}{s} = 1. \tag{4.C}$$

This is equivalent to the condition

$$\frac{1}{n+1} \sum_{k=0}^{n} (k+1)^{1/s} |y_k| = O(1). \tag{4.D}$$

Similarly, a sequence y in dv_r satisfies the condition (2.E) if and only if

$$\sum_{2^n} |y_{k-1}| = o(2^{n/s}), \qquad \frac{1}{r} + \frac{1}{s} = 1, \tag{4.E}$$

which is equivalent to

$$\frac{1}{n+1} \sum_{k=0}^{n} (k+1)^{1/s} |y_k| = o(1). \tag{4.F}$$

5. Multiplier Results

If E and F are FK-spaces, we write

$$E \cdot F = \{x \cdot y := (x_k y_k) \mid x \in E,\ y \in F\},$$
$$(E \to F) = \{y = (y_k) \mid x \cdot y \in F \text{ for all } x \in E\}.$$

The set $E \cdot F$ need not be a linear space. The set $(E \to F)$ is a sequence space but it need not be an FK-space. However, if E and F are BK-spaces, then $(E \to F)$ is a BK-space under the norm $\|y\| = \sup_{\|x\|_E \leqslant 1} \|x \cdot y\|_F$.

An FK-space E containing ϕ has the property AB if and only if $E = bv \cdot E$ and it has the property AK if and only if $E = (bv \cap c_0) \cdot E$ [10]. Similarly E has the property σB if and only if $E = q \cdot E$ and it has the property σK if and only if $E = (q \cap c_0) \cdot E$ [3]. Now we show that strong Cesàro summability and strong Cesàro boundedness for an FK-space are also equivalent to multiplier statements.

(5.1) Theorem. *Let E be an FK-space containing ϕ with a defining family of continuous seminorms $p^1 \leqslant p^2 \leqslant p^3 \leqslant \cdots$. Let $A_N \subset E'$ such that $p^N(x) = \sup_{f \in A_N} |f(x)|$ for all $x \in E$. For $1 \leqslant r < \infty$ the following statements are equivalent*:

(a) $$x \in E_{[\sigma B]_r};$$

(b) $$\sup_n \sup_{f \in A_N} \frac{1}{n+1} \sum_{k=0}^{n} |f(s^k x)|^r < \infty \qquad \text{for all} \quad N;$$

(c) $$dv_r \cdot x \subset E.$$

Proof. (a) ⇒ (b). This is immediate. (b) ⇒ (c). Suppose (b) and let $y \in dv_r$. We show $y \cdot x \in E$ by showing that $\sigma^t(y \cdot x) = (1/(t+1)) \sum_{k=0}^{t} s^k(y \cdot x)$ is a Cauchy sequence in E for $t = 2^m$. Using summation by parts we obtain $\sigma^t(y \cdot x) = \sum_{k=0}^{t} \{(1 - k/(t+1))\, \Delta y_k + (1/(t+1))\, y_{k+1}\}\, s^k x$. For $s = 2^m < 2^n = t$ and $f \in A_N$ we have

$$\begin{aligned}
&|f(\sigma^t(y \cdot x) - \sigma^s(y \cdot x))| \\
&\quad \leqslant \left(\frac{1}{s+1} - \frac{1}{t+1}\right) \sum_{k=0}^{s} |k \Delta y_k - y_{k+1}|\, |f(s^k x)| \\
&\qquad + \sum_{k=s+1}^{t} \left| \left(1 - \frac{k}{t+1}\right) \Delta y_k + \frac{1}{t+1}\, y_{k+1} \right| |f(s^k x)| \\
&\quad \leqslant \frac{1}{2^m} \sum_{j=0}^{m} \Sigma_{2^j} (k |\Delta y_{k-1}| + |y_k|)\, |f(s^k x)| \\
&\qquad + \sum_{j=m+1}^{n} \Sigma_{2^j} \left(\left(1 - \frac{k}{t+1}\right) |\Delta y_{k-1}| + \frac{1}{t+1} |y_k| \right) |f(s^k x)|.
\end{aligned}$$

Using Hölder's inequality on the sums $\sum_{2^j}$ we obtain

$$\frac{1}{2^m}\sum_{j=0}^{m}\{\textstyle\sum_{2^j}(k|\Delta y_{k-1}|+|y_k|)^{p'}\}^{1/p'}\{\sum_{2^j}|f(s^kx)|^p\}^{1/p}$$
$$+\sum_{j=m+1}^{n}\left\{\textstyle\sum_{2^j}\left(\left(1-\frac{k}{t+1}\right)|\Delta y_{k-1}|+\frac{1}{t+1}|y_k|\right)^{p'}\right\}^{1/p'}$$
$$\times\{\textstyle\sum_{2^j}|f(s^kx)|^p\}^{1/p}$$
$$\leqslant\frac{1}{2^m}\sum_{j=0}^{m}\{2^{j+1}(\textstyle\sum_{2^j}|\Delta y_{k-1}|^{p'})^{1/p'}+\max_{2^j}|y_k|\}\,2^{(j+1)/p}p_r^N(x)$$
$$+\sum_{j=m+1}^{n}\left\{(\textstyle\sum_{2^j}|\Delta y_{k-1}|^{p'})^{1/p'}+\frac{2^{j/p'}}{t+1}\max_{2^j}|y_k|\right\}2^{(j+1)/p}p_r^N(x)$$
$$\leqslant p_r^N(x)\left\{\frac{2^{M+2}}{2^m}\left(\sum_{j=0}^{M}2^{j/p}(\textstyle\sum_{2^j}|\Delta y_{k-1}|^{p'})^{1/p'}+\sup|y_k|\right)\right.$$
$$+\left(4\sum_{j=M+1}^{m}2^{j/p}(\textstyle\sum_{2^j}|\Delta y_{k-1}|^{p'})^{1/p'}+\sup_{k>2^M}|y_k|\right)$$
$$\left.+\left(\sum_{j=m+1}^{n}2^{j/p}(\textstyle\sum_{2^j}|\Delta y_{k-1}|^{p'})^{1/p'}+2\sup_{k>2^M}|y_k|\right)\right\}.$$

This can be made arbitrarily small by choosing M and m sufficiently large. Thus $p^N(\sigma^t(y\cdot x)-\sigma^s(y\cdot x))=\sup_{f\in A_N}|f(\sigma^t(y\cdot x)-\sigma^s(y\cdot x))|\to 0$ as $n, m\to\infty$. Since E is complete and has continuous coordinate functionals, $\sigma^t(y\cdot x)$ converges to $y\cdot x$. This shows $dv_r\cdot x\subset E$.

(c)$\Rightarrow$(a). Suppose $dv_r\cdot x\subset E$. Then by the closed graph theorem, $T_x(y):=y\cdot x$ is a continuous map from dv_r to E [14]. Let p be a continuous seminorm on E. Then $p\circ T_x$ is a continuous seminorm on dv_r. Thus $p(T_x(y))\leqslant K_p\|y\|_{dv_r}$ for some constant K_p. Hence for $f\in E'$ with $|f|\leqslant p$ we have $f\circ T_x\in dv_r'$ and $|f(T_x(y))|\leqslant p(T_x(y))\leqslant K_p\|y\|_{dv_r}$. Since dv_r has the property $[\sigma B]_r$, we have

$$\sup_{|f|\leqslant p}\frac{1}{n+1}\sum_{k=0}^{n}|f(s^n(x\cdot y))|^r$$
$$=\sup_{|f|\leqslant p}\frac{1}{n+1}\sum_{k=0}^{n}|f\circ T_x(s^ny)|^r<\infty.$$

This shows that every sequence in $dv_r\cdot x$ has the property $[\sigma B]_r$. ∎

(5.2) Theorem. *Let E be an FK-space containing ϕ and let $1\leqslant r<\infty$. Then E has the property $[\sigma B]_r$ if and only if $E=dv_r\cdot E$.*

Proof. If E has the property $[\sigma B]_r$, then by (5.1) (a) $\Rightarrow$ (c) we have $dv_r \cdot E \subset E$. Since the sequence (1, 1, ...) is in dv_r, the opposite inclusion is immediate. The converse follows from (5.1) (c) $\Rightarrow$ (a). ∎

(5.3) THEOREM. *Let E be an FK-space containing ϕ and let $1 \leqslant r < \infty$. Then E has the property $[\sigma K]_r$ if and only if $E = (dv_r \cap c_0) \cdot E$.*

Proof. Suppose E has the property $[\sigma K]_r$. Then E has the property σK. It was shown in [3] that if E has σB then $E_{\sigma K} = (q \cap c_0) \cdot E$. For the same reasons $(dv_r)_{[\sigma K]_r} = (dv_r)_{\sigma K} = (q \cap c_0) \cdot dv_r = dv_r \cap c_0$. Thus since $E = dv_r \cdot E$ by (5.2), we have $E = E_{\sigma K} = (q \cap c_0) \cdot E = (q \cap c_0) \cdot (dv_r \cdot E) = (dv_r \cap c_0) \cdot E$. Conversely suppose $E = (dv_r \cap c_0) \cdot E$. For each $x \in E$, $T_x(y) = y \cdot x$ is a continuous map from $dv_r \cap c_0$ into E. Let p be a continuous seminorm on E. As in the proof of (5.1), there exists a constant K_p such that $\sup_{|f| \leqslant p} (1/(n+1)) \sum_{k=0}^{n} |f(s^n(x \cdot y) - x \cdot y)|^r = \sup_{|f| \leqslant p} (1/(n+1)) \sum_{k=0}^{n} |f \circ T_x(s^n y - y)|^r \leqslant \sup_{|g| \leqslant K_p \|\cdot\|_{dv_r}} (1/(n+1)) \sum_{k=0}^{n} |g(s^n y - y)|^r$. Since $dv_r \cap c_0$ has the property $[\sigma K]_r$, this tends to 0. ∎

The following is a consequence of (3.4), (3.5), and (5.2).

(5.4) PROPOSITION. *Let E be an FK-space containing ϕ and let $1 \leqslant r < \infty$. Then the space $E_{[\sigma B]_r}$ is an FK-space having the property $[\sigma B]_r$.*

(5.5) *Remark.* Multiplier statements corresponding to (5.2) (respectively (5.3)) do not hold for spaces satisfying condition (2.F) (respectively condition (2.E)) with respect to the multipliers dv_r (respectively $dv_r \cap c_0$) or with respect to the sequences satisfying conditions (4.C) (respectively condition (4.E)). The spaces dv_r and H_r serve as counterexamples. However, we can obtain a partial result which we give without proof.

(5.6) PROPOSITION. *Let E be an FK-space containing ϕ, let $1 \leqslant r < \infty$, and let x be a sequence. If $x \cdot y \in E$ for every sequence y in dv_r satisfying condition* (2.E), *then the sequences $x \cdot y$ satisfy* (2.F).

6. FUNCTION SPACES

We now consider spaces of 2π-periodic functions or distributions g for which Fourier coefficients $\hat{g}(k)$ are defined [8]. Sequences will be defined on the integers, and the sequences in q, dv_r, and bv will be assumed to be symmetric (that is, $y_k = y_{-k}$). Here e^k is the function $e^k(x) = e^{ikx}$ and $s^n g(x) = \sum_{|k| \leqslant n} \hat{g}(k) e^{ikx}$.

Zygmund [15, Theorem XIII.7.3] shows that for every function g in $C_{2\pi}$ and $1 \leqslant r < \infty$ we have

$$\frac{1}{n+1} \sum_{k=0}^{n} |s^k g(x) - g(x)|^r \to 0 \qquad \text{uniformly for all } x. \tag{6.A}$$

Since the norm on $C_{2\pi}$ is $\|g\|^\infty = \sup_x |g(x)|$, this is equivalent to saying that $C_{2\pi}$ has the property $[\sigma K]_r$, where $A_{\|\cdot\|} = \{F_x \in C'_{2\pi} \mid F_x(g) := g(x), 0 \leqslant x \leqslant 2\pi\}$ as defined by (2 .A). This shows by (5.2) and (5.3) that for all $1 \leqslant r < \infty$, we have

$$dv_r \cdot \hat{C}_{2\pi} = \hat{C}_{2\pi} = (dv_r \cap c_0) \cdot \hat{C}_{2\pi}. \tag{6.B}$$

Since $(\hat{C}_{2\pi} \to \hat{C}_{2\pi}) = (\hat{L}^1_{2\pi} \to \hat{L}^1_{2\pi}) = (\hat{M}_{2\pi} \to \hat{M}_{2\pi}) = (\hat{L}^\infty_{2\pi} \to \hat{L}^\infty_{2\pi}) = \hat{M}_{2\pi}$ [8, Vol. 2, p. 246], an immediate consequence is the result $dv_r \subset \hat{M}_{2\pi}$ for all $1 \leqslant r < \infty$ [6]. We also obtain Fomin's integrability result $dv_r \cap c_0 = (dv_r)_{AD} \subset (\hat{M}_{2\pi})_{AD} = \hat{L}^1_{2\pi}$ [6, 9]. Since $e = (..., 1, 1, 1, ...) \in dv_r$, we have also $dv_r \cdot \hat{L}^1_{2\pi} = \hat{L}^1_{2\pi}$, $dv_r \cdot \hat{M}_{2\pi} = \hat{M}_{2\pi}$, and $dv_r \cdot \hat{L}^\infty_{2\pi} = \hat{L}^\infty_{2\pi}$.

Conversely, our multiplier results show that (6.A) can be obtained from Fomin's integrability result.

Furthermore, $(\hat{M}_{2\pi})_{AD} = \hat{L}^1_{2\pi}$ and $(\hat{L}^\infty_{2\pi})_{AD} = \hat{C}_{2\pi}$. By (5.2), (3.2), and (5.3) we have the following.

(6.1) THEOREM. *Let $1 \leqslant r < \infty$. The spaces $C_{2\pi}$ and $L^1_{2\pi}$ have the property $[\sigma K]_r$. The spaces $L^\infty_{2\pi}$ and $M_{2\pi}$ have the property $[\sigma B]_r$.*

(6.2) *Remark.* Theorem 6.1 for $L^1_{2\pi}$ is stronger than the theorem of Fejér's which states for $f \in L^1_{2\pi}$,

$$\frac{1}{n+1} \left\| \sum_{k=0}^{n} (s^k f - f) \right\|_{L^1} = o(1) \qquad (n \to \infty).$$

This is equivalent to

$$\sup_{F \in A} \frac{1}{n+1} \left| \sum_{k=0}^{n} F \cdot (s^k f - f) \right| = o(1) \qquad (n \to \infty)$$

for some subset A of the dual of $L^1_{2\pi}$. Since the dual of $L^1_{2\pi}$ is $L^\infty_{2\pi}$ and the continuous linear functionals on $L^1_{2\pi}$ are of the form $F_g(f) = \int_0^{2\pi} g \cdot f$ for $g \in L^\infty_{2\pi}$, we have

$$\sup_{\|g\|^\infty \leqslant 1} \frac{1}{n+1} \left| \sum_{k=0}^{\infty} \int_0^{2\pi} g \cdot (s^k f - f) \right| = o(1) \qquad (n \to \infty).$$

Theorem 6.1 shows that the absolute value can be taken inside the summation and raised to any power $1 \leqslant r < \infty$ to obtain

$$\sup_{\|g\|^{\infty} \leqslant 1} \frac{1}{n+1} \sum_{k=0}^{n} \left| \int_0^{2\pi} g \cdot (s^k f - f) \right|^r = o(1) \qquad (n \to \infty). \tag{6.C}$$

One could consider direct proofs of (6.C) from (6.A) but the main idea here is the equivalence of convergence theorems and multiplier theorems.

(6.3) *Remark.* The following example due N. Tanović-Miller shows that (6.C) cannot be further strengthened by taking the supremum inside the summation. That is, the property $[\sigma K]_r$ cannot be strengthened to the property (2.E). The example shows that for each $1 \leqslant r < \infty$, there exist $f \in L^1_{2\pi}$ such that

$$\frac{1}{n+1} \sum_{k=0}^{n} \sup_{\|g\|^{\infty} \leqslant 1} \left| \int_0^{2\pi} g \cdot (s^k f - f) \right|^r$$
$$= \frac{1}{n+1} \sum_{k=0}^{n} \|s^k f - f\|_{L^1}^r \neq O(1) \qquad (n \to \infty).$$

It is sufficient to let $r = 1$, since by Hölder's inequality

$$\frac{1}{n+1} \sum_{k=0}^{n} \|s^k f - f\|_{L^1} \leqslant \left(\frac{1}{n+1} \sum_{k=0}^{n} \|s^k f - f\|_{L^1}^r \right)^{1/r}.$$

Consider the cosine series $(1/2)\, a_0 + \sum_{k=1}^{\infty} a_k \cos kx$, where $a_0 = a_1 = 0$, and $a_k = 1/\sqrt{\log k}$, $(k \geqslant 2)$. We have $a_k \downarrow 0$, $k\Delta a_k \to 0$, and $\sum_k (k+1)\, |\Delta^2 a_k| < \infty$ since $\Delta a_k \sim 1/k \log^{3/2} k$, and $\Delta^2 a_k \sim 1/k^2 \log^{3/2} k$. By a classical result of Kolmogorov [7, Vol. 1, 7.3.1 and 7.3.2] the cosine series converges to

$$f(x) = \frac{1}{2} \sum_{k=0}^{\infty} (k+1)\, \Delta^2 a_k F_k(x) \tag{6.D}$$

(F_k denotes the Fejér kernel) pointwise for $x \neq 0 \pmod{2\pi}$, $f \in L^1_{2\pi}$, and (6.D) is the Fourier series of f (moreover, $f \geqslant 0$). By partial summation and by (6.D), we have

$$s^n f(x) - f(x) = \tfrac{1}{2} n \Delta a_{n-1} F_{n-1}(x) + \tfrac{1}{2} a_n D_n(x)$$
$$- \tfrac{1}{2} \sum_{k=n-1}^{\infty} (k+1)\, \Delta^2 a_k F_n(x)$$

(D_n denotes the Dirichlet kernel) for $x \neq 0 \pmod{2\pi}$. Thus

$$\|s^n f - f\|_{L^1} \geq \tfrac{1}{2}|a_n| \, \|D_n\|_{L^1} - \tfrac{1}{2}n|\Delta a_{n-1}| \, \|F_n\|_{L_1} - \tfrac{1}{2} \sum_{k=n-1}^{\infty} (k+1) \, |\Delta^2 a_k| \, \|F_n\|_{L^1}.$$

Since $\|F_n\|_{L^1} = 1$, given $\varepsilon > 0$, there exists N such that for $n > N$,

$$\|s^n f - f\|_{L^1} \geq \frac{1}{2} |a_n| \, \|D_n\|_{L^1} - \frac{\varepsilon}{2}.$$

Hence $(1/(n+1)) \sum_{k=0}^{n} \|s^n f - f\|_{L^1} \geq (1/(n+1)) \sum_{k=N}^{n} |a_k| \, \|D_k\|_{L^1} - \varepsilon/2$. But $\|D_k\|_{L^1} \sim (4/\pi^2) \log k$, and consequently $(1/(n+1)) \sum_{k=0}^{n} \|s^k f - f\|_{L^1} \to \infty$, $(n \to \infty)$.

Since $L^\infty_{2\pi} = (L^\infty_{2\pi})_{\sigma B}$ and $M_{2\pi} = (M_{2\pi})_{\sigma B}$, we have

$$L^\infty_{2\pi} = (C_{2\pi})_{[\sigma B]_r} \qquad \text{and} \qquad M_{2\pi} = (L^1_{2\pi})_{[\sigma B]_r}. \tag{6.E}$$

From (3.6) and the first identity in (6.E) we obtain the following.

(6.4) THEOREM. *Let $g \in L^1_{2\pi}$ and $1 \leq r < \infty$. Then $g \in L^\infty_{2\pi}$ if and only if*

$$\|g\|^\infty_r = \sup_{n,x} \left\{ \frac{1}{n+1} \sum_{k=0}^{n} |s^k g(x)|^r \right\}^{1/r} < \infty.$$

Furthermore $\|\cdot\|^\infty_r$ is a defining norm on $L^\infty_{2\pi}$.

We can obtain a similar result for the space $M_{2\pi}$ from the second identity in (6.E).

Since the continuous linear functionals on $L^1_{2\pi}$ are of the form $F_f(g) = \int_0^{2\pi} f \cdot g$ for $f \in L^\infty_{2\pi}$, we have $\|g\|^1_r = \sup_{\|f\|^\infty \leq 1} \sup_n \{(1/(n+1)) \sum_{k=0}^{n} |\int_0^{2\pi} s^k(f \cdot g)|^r\}^{1/r}$ for $g \in L^1_{2\pi}$. Consequently we obtain the following.

(6.5) THEOREM. *For each $g \in L^1_{2\pi}$ and $1 \leq r < \infty$ we have*

$$\sup_{\|f\|^\infty \leq 1} \sup_n \frac{1}{n+1} \sum_{k=0}^{n} \left| \int_0^{2\pi} s^k(f \cdot g) \right|^r < \infty.$$

Finally, since $(\hat{L}^\infty_{2\pi})^\varphi = (\hat{C}_{2\pi})^\varphi = \hat{M}_{2\pi}$ and $(\hat{M}_{2\pi})^\varphi = (\hat{L}^1_{2\pi})^\varphi = \hat{L}^\infty_{2\pi}$ we obtain the following from (3.4).

(6.6) THEOREM. *For each $1 \leq r < \infty$, $\hat{L}^\infty_{2\pi} = (\hat{L}^1_{2\pi} \to B_r)$ and $\hat{M}_{2\pi} = (\hat{L}^\infty_{2\pi} \to B_r)$.*

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